SOME APPLICATIONS OF PETTERSSEN'S VELOCITY EQUATIONS WITH REFERENCE TO AIRWAY FORECASTING

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A largely qualitative discussion and interpretation of equations appearing in "Kinematical and Dynamical Properties of the Field of Pressure with Application to Weather Forecasting" appears in the "Practical Rules for Prognosticating the Movement and Development of Pressure Centers", by the same author. It is therefore intended to confine this discussion in so far as possible to a quantitative application of the equations of velocity.

Two methods of procedure are available - (1) A discussion of a series of maps and (2) A general discussion of the equations and a shorter series of maps. The latter is thought preferable as a general discussion of the equations must apply in all cases.

The numerical solution of these equations presupposes a correctly analized map and when taken together with the practical rules of the above mentioned publication improves noticeably with continued practice. When the data available to work with are considered it should not be expected that the solution obtained will be exact at any time--in fact, to be so, should rather be classed as accidental. A proper criterion for the value of the numerical solution of the equations depends entirely on how nearly accurate they foretell the movements of the element considered.

Pressure and its derivatives are used in the development of the equations. The pressure appearing on a weather map is sea level pressure--the result of a reduction through a fictitious air column from the barometer to sea level. This may lead to unimportant errors in pressure gradients where the stations are within a few thousand feet of sea level and more serious errors in the mountains. A more important source of error, because of its importance in the equations, is that which must be used for the instantaneous pressure variations--the 3-hourly pressure tendency. This gives, instead of an instantaneous slope of the barogram, an average slope over a period of 3 hours. Very often the diurnal variation in pressure, superimposed on the barogram, that is due to the existing field of pressure, causes changes in the slope of the barogram such that a 3-hour average is of little value. Many of the local irregularities in pressure tendencies may be corrected by eliminating irregularities in the isallobars. The equations will have little or no value in mountainous regions because (1) frontal zones are apt to be extremely irregular, (2) pressure tendencies, while they may be representative of frontal passage, are not comparable with one another due to large differences in station elevation, (3) pressure gradients too often are not true.

The fundamentals of the equations involved follows.

ISOBARS

The equation for the velocity of an isobar, C_i, is given by

$$C_i = - \frac{\frac{\partial p}{\partial t}}{\frac{\partial p}{\partial x}}$$

where $\frac{\partial p}{\partial t}$ is the pressure tendency T, and $\frac{\partial p}{\partial x}$ is the pressure ascendant, usually indicated as 1/h where h is the distance between two neighboring unit isobars. This equation states that the velocity of an isobar is proportional to the pressure tendency and inversely proportional to the pressure ascendant. Pressure tendencies are reported in hundredths of an inch and isobars separated by 0.1 inch. In order to get the numerator and denominator in the same units it is convenient to write the pressure ascendant at 10/h and eliminate the decimal point in both the numerator and denominator. The above equation then becomes

$$C_i = -\frac{Th}{10}$$

In solving this equation T and h are available from a single weather map. It is necessary that isobars be carefully drawn in the determination of h and convenient to choose the X axis normal to the isobar in question, remembering that $\frac{\partial p}{\partial x}$ being the pressure ascendant, is directed toward higher pressure. If the positive end of the X axis is directed toward lower pressure the denominator of the above equation must carry a minus sign.

In using this equation points along an isobar should be selected for computation where the pressure tendencies are fairly uniform. Rapidly changing pressure tendencies with distance results in a doubtful determination of the value of T, especially so, as an isobar may be slightly in error with the most careful drawing.

From this equation it is possible to determine likely changes in the pressure field and tendencies of the pressure gradient to increase or decrease during a particular interval. Isobars, neglecting acceleration, will vary from the exact determined location for periods of less than 24 hours due to the influence of diurnal pressure variation.

TROUGHS, WEDGES AND PRESSURE CENTERS

For practical purposes the equation for the velocity of trough lines, wedge lines and pressure centers, C_x, is given by

$$C_x = -\frac{\frac{\partial^2 p}{\partial x \partial t}}{\frac{\partial^2 p}{x^2}} = -\frac{p_{101}}{p_{200}}$$

where C_x is normal to the trough or wedge line, $\frac{\partial^2 p}{\partial x \partial t}$ is the component of the isallobaric ascendant, $\frac{\partial^2 p}{\partial x^2}$ is the curvature of the pressure profile and the subscripts indicate the order of the derivative and the independent variables involved. The coefficients p_{101} and p_{200} are evaluated by choosing an X axis normal to the wedge or trough line and an arbitrary unit and half unit of length on each side of the trough or wedge line. Then the equation

may be written in the practical form for numerical solution

$$C_x = -\frac{T^{\frac{1}{2},0} - T^{-\frac{1}{2},0}}{(p^{1,0} - p^{0,0}) + (p^{-1,0} - p^{0,0})}$$

or when the trough is too narrow for determination of the tendency values at $\frac{1}{2},0$ and $-\frac{1}{2},0$

$$C_x = -\frac{\frac{1}{2}\,(T^{1,0} - T^{-1,0})}{(p^{1,0} - p^{0,0}) + (p^{-1,0} - p^{0,0})}$$

where

$$T^{\frac{1}{2},0} - T^{-\frac{1}{2},0} = p_{101}$$

$$(p^{1,0} - p^{0,0}) + (p^{-1,0} - p^{0,0}) = p_{200}$$

The unit of length can be taken convenient to each particular case but must be such that the values for T and p are the result of the trough or wedge and represent the closest approximation to the true differentials. Barometric tendencies, carrying more weight in the equation than pressure, should determine the units. Small changes in the curvature of the pressure profile will affect results but little.

In computing the movement of a pressure center it is desirable to have a component in two directions. For this purpose coordinate axes are set up with origin as near the pressure center as possible. The equation for the velocity along the Y axis, C_y, is identical to that along the X axis

$$C_y = -\frac{T^{\frac{1}{2},0} - T^{-\frac{1}{2},0}}{(p^{1,0} - p^{0,0}) + (p^{-1,0} - p^{0,0})}\,.$$

It is not at all necessary that the units along both axes be the same. The distance computed along each axis may be considered as a displacement vector, being a component of a rectangular resolution of an unknown vector the terminus of which defines the future position of the pressure center for any desired time interval. Then in the case where the axes are not normal to each other the unknown vector defining the pressure center cannot be found by the well known parallelogram law for vectors but must be found by drawing perpendiculars from the terminus of each known vector. The terminus of the unknown vector is the point of intersection of the two perpendiculars thus drawn.

FRONTS

The equation for the velocity of a front, C_f, is given by

$$C_f = -\frac{\frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t}}{\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x}}$$

where the subscript 1 denotes a point immediately ahead of a front, 2 a point exactly opposite and immediately behind the front and as before,

$\frac{\partial p}{\partial t}$ is the pressure tendency and $\frac{\partial p}{\partial x}$ is the pressure ascendant. In open warm sector cyclones it is usually convenient to choose a warm sector isobar as the X axis. Then at a warm front $\frac{\partial p_2}{\partial x}$ is zero and at a cold front $\frac{\partial p_1}{\partial x}$ is zero. Where neither the $\frac{\partial p_1}{\partial x}$ or $\frac{\partial p_2}{\partial x}$ can be eliminated by choosing the X axis along an isobar it is convenient and easy to write one term of the denominator in terms of the other. The denominator of this equation is always positive.

The difficulty in the application of this equation is readily seen to be in determining values in the numerator. With a fairly dense net work of stations, such as exist along airways, and eliminating local variations in the tendencies by smoothing irregularities in the isallobars, $\frac{\partial p_1}{\partial t}$ can be determined with reasonable accuracy but the 3-hourly tendencies at stations immediately behind a front are of no value in determining the slope of the barogram. They have been subjected, for the greater part of the 3 hour period, to pressure variations ahead of the front.

Because of this difficulty better results are usually obtained by using the equations discussed under "Troughs, Wedges, and Pressure Centers". Most all trough lines are "Fronts", the principal difference between the two being that there must not be an isallobaric discontinuity along a trough line while an isallobaric discontinuity **is essential** to **the existence** of a front. To carry out the numerical differentiation called for in the velocity of a trough line is, of course, mathematically incorrect due to the discontinuity at the front. From the practical viewpoint we may, in many cases, smooth out this discontinuity of the isallobaric profile at the front and use the equation for the velocity of a trough. This process is permissible only if by doing so there is retained a tendency profile characteristic of a trough and values for $T^{\frac{1}{2},0}$ and $T^{-\frac{1}{2},0}$ are not affected. This usually happens when the magnitude of the discontinuity is small. In all cases where the magnitude is large this process will lead to large errors. Figure I illustrates a tendency profile across a front where the discontinuity may reasonably be eliminated and the equation for a trough used.

GENERAL APPLICATION

Many cases appear on a weather chart where numerical solutions fail completely. Such cases are usually apparent before computation upon close study of the field of pressure and isallobars. When the isallobaric discontinuity at a front is large and, as usually happens, representative values for the pressure tendency behind the front are lacking, there is no method for the numerical solution of the equation.

Where the isallobaric ascendant about a trough is small, say ($T^{\frac{1}{2},0} - T^{-\frac{1}{2},0} = .02$) computations will fail by as much as a computed unit when there is allowed a tendency error of only .01 inch at $T^{\frac{1}{2},0}$ and $T^{-\frac{1}{2},0}$. Again if pressure gradients are small the denominator of the equations for troughs and fronts cause considerable errors. Many fronts, involving both these difficulties, cross Texas from west to east proceeding a southward advance of a cold front.

In applying Petterssen's equations several points should be selected along a trough or front and the line connecting the computed displacements should be smooth. In this way possible irregularities in the computed movement are eliminated. If all the points lie on a smooth line confidence in the computation is reasonable.

In the case of fronts having constant consecutive 24 hour movements it is still possible to have a diurnal change in velocity due to different pressure variations in the different air masses. One important case of this is a cyclone moving across the northern plains states where Pacific air which has crossed the mountains and is relatively warm and dry is behind the front. This air is usually accompanied by a quite large diurnal pressure variation. This variation in the velocity is important in airway forecasting and can be allowed for only empirically as a departure from the computed displacement.

MAPS

The Central Section Airway map was used for this study. Plotting and symbols are about the same as is used for airway purposes, with the exception that wind velocities are plotted on the half Beaufort scale, and cloud data not entered. Besides the wind there is plotted on the map both temperature and dewpoint, sky conditions, weather and pressure characteristics For the fronts solid lines are cold fronts and dashed lines warm fronts. A few of the isallobars are indicated by short dashed lines.

MAP 1

This map was selected to illustrate the application of velocity equations for the movement of the front extending from Lake Superior south-southwestward to Oklahoma, then northwest into western Colorado with a low centered on the front in southwestern Oklahoma. West of the low center no application of the velocity equations was feasible because of the isallobaric discontinuity occurring at the front and the impossibility of determining pressure characteristics immediately behind the front for use in the front formula. Also, no computations were made for the front extending from the center of the low southward through Texas, the isallobaric ascendant being too small. Computation of the movement of the cold front from Wisconsin southwestward to the center of the low is possible at the places indicated on the map by four axes drawn normal to the front. At these points the

trough formula may be used as there is not a large discontinuity in the isallobars at the front. In each case the pressure at the front is indicated, the tendency values at the half unit are indicated along the axes, and the pressures at the full unit interpolated from the isobars. For the most northern axis a large unit could be used as tendency values ahead of the front in this region are all ±2 and for the axis through northeastern Oklahoma a short unit was necessary to obtain proper values for the tendencies at the half units; it will be noticed that tendency values at Little Rock, Texarkana and Shreveport all are positive. Substituting the values indicated on the map in the trough formula the movement for 6 and 12 hours is indicated by the letter "C" along the axes. Solid lines east of the present position of the front indicate the actual movement, while the dashed lines indicate the computed movement. It may be noticed that, for the 6 hour period the agreement between computed and actual movements in the south is very good, while in the north the displacement of the front is approximately one-half of the computed displacement. On the other hand, for the 12 hour period, the actual movement compared to the computed movement agrees very well, with the greatest difference between computed and actual movements occurring in Arkansas. In any case, with the possible exception of the 6 hour movement for the northern half of the front the agreement between computed and actual displacements is better than an estimated displacement would be.

No attempt was made to compute the movement of the warm front numerically. The trough accompanying the warm front is too indefinite for application of the velocity equations.

MAP 2

Map 2 was selected because of the double cold front structure. On examining the isallobars about the low pressure center it seems apparent that the isallobaric ascendant is directed northwestward from the warm front, or from a point very near to the warm front. Strictly, this should limit the length of the axes to such an extent that the equation for the movement of pressure center could not be used due to the end points being too close to the fronts. It will be noticed that an East West axis would cut the warm front. This being undesirable, the direction of movement of the low pressure center was assumed to be approximately that of the direction of the warm sector isobars. This direction line is indicated on the map by the arrow extending north northeastward from the low pressure center. In order to determine the magnitude of the movement a north south axis was selected in such a way as to get approximately correct value for the pressure tendencies. North of the pressure center, not far from the -6 isallobar a half unit was taken, and from the length of the half unit the full unit was extended in both directions. This gives a value of the pressure tendencies at the half unit of very nearly -.06, and at the $-\frac{1}{2}$ unit of +.04, pressure values at the full unit being 29.46 and 29.38. The tendencies between the two cold fronts are, as might be expected from the number of stations available, rather difficult to interpret; however, as near to the center of the low as the $-\frac{1}{2}$ unit is and in view of the -.06 at Sioux City, and the required discontinuity at the front immediately west of Omaha, +.04 at the $-\frac{1}{2}$ unit seems reasonable. Substituting these values in the trough formula we get a 6-hourly movement

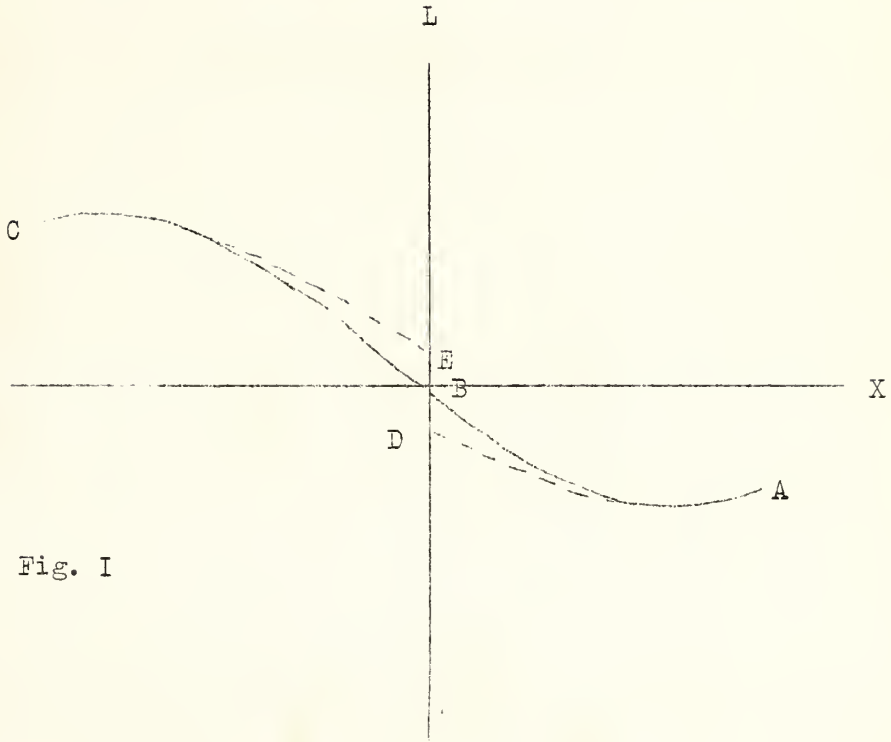


Fig. I

L is a trough line or front.
ABC - isallobaric profile through a trough.
AD-EC - isallobaric profile through a front.
DE - magnitude of discontinuity at a front.

VELOCITY OF ISOBAR

$$C_i = -\frac{\frac{\partial p}{\partial t}}{\frac{\partial p}{\partial x}} = -\frac{Th}{10}$$

$\frac{\partial p}{\partial t}$ = pressure tendency T

$\frac{\partial p}{\partial x}$ = pressure ascendant $\frac{1}{h}$

VELOCITY OF TROUGH, WEDGE OR PRESSURE CENTER

$$C_x \text{ or } C_y = -\frac{\frac{\partial^2 p}{\partial x \partial t}}{\frac{\partial^2 p}{\partial x}} = -\frac{T^{\frac{1}{2},0} - T^{-\frac{1}{2},0}}{(p^{1,0}-p^{0,0})+(p^{-1,0}-p^{0,0})}$$

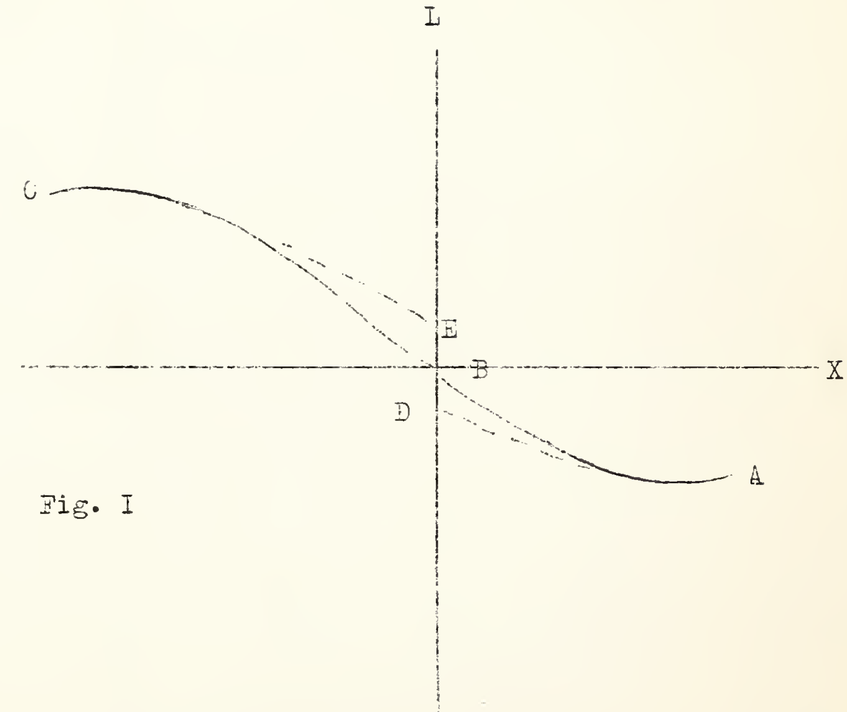
$\frac{\partial^2 p}{\partial x \partial t}$ = x component of the isallobaric profile

$\frac{\partial^2 p}{\partial x}$ = curvature of the pressure profile

VELOCITY

$$C_f = \frac{\frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t}}{\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x}}$$

Subscript 1 denotes a point immediately ahead of a front, and 2 a point exactly opposite and immediately behind the front.



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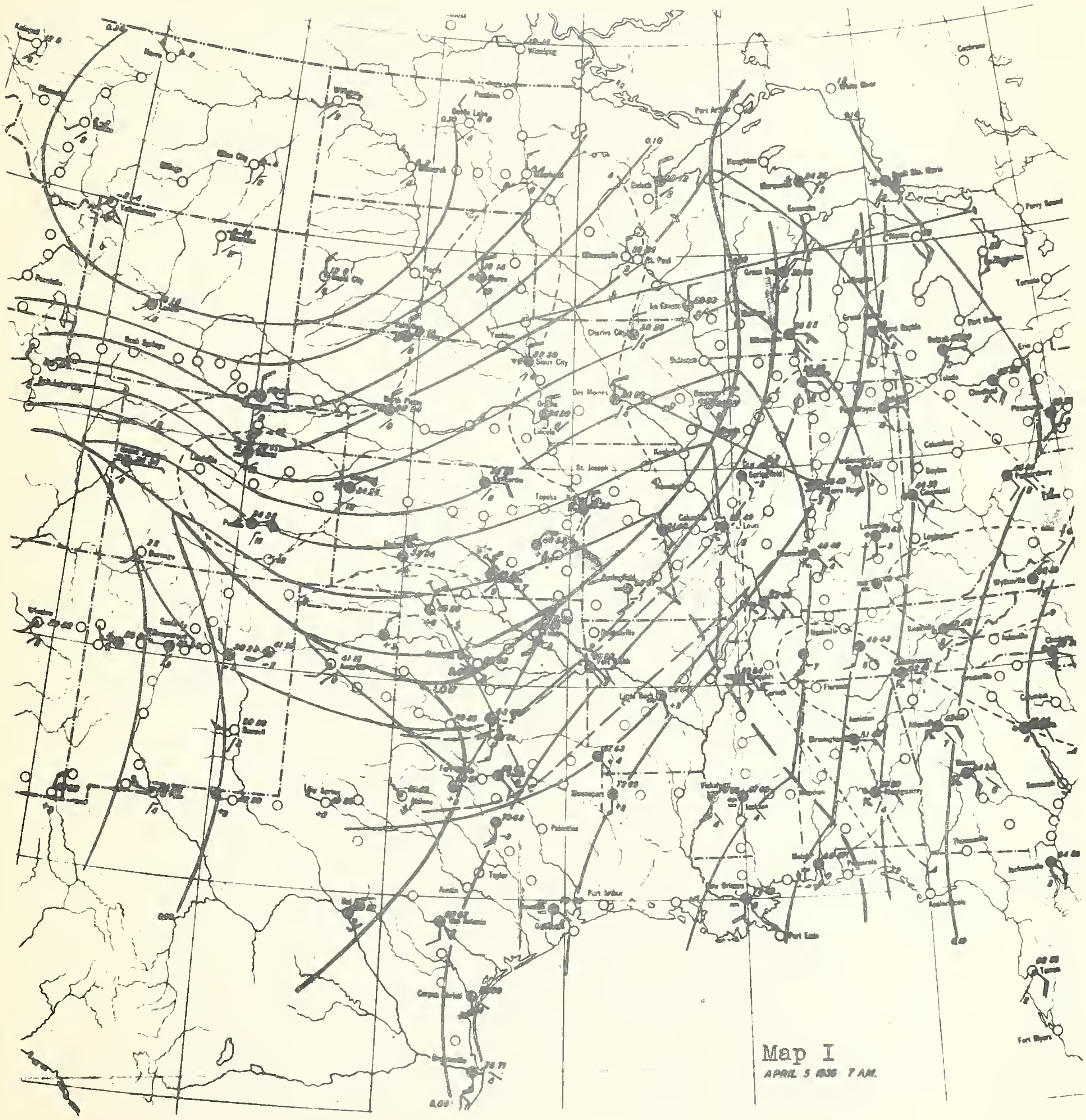

Map I
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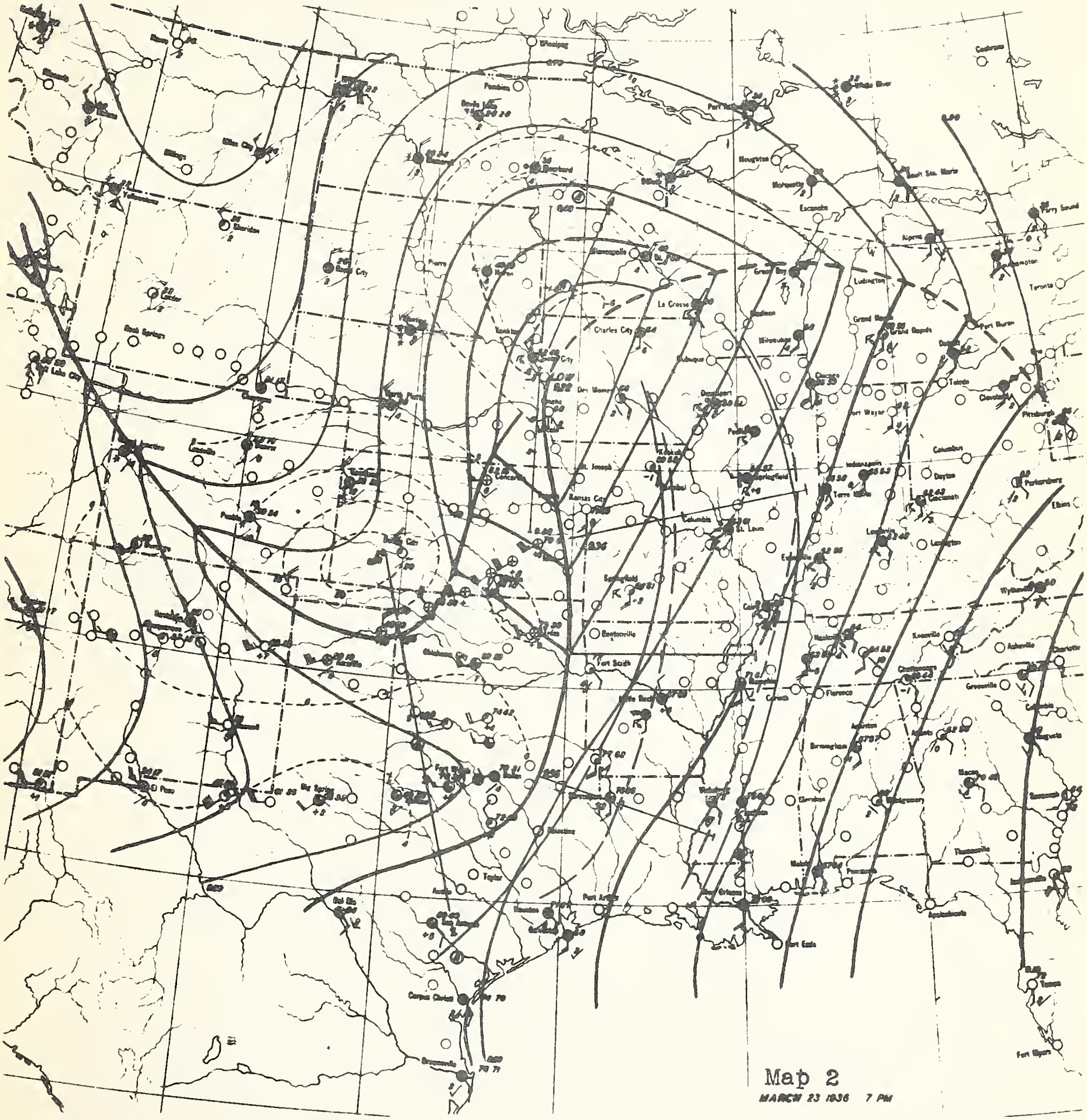

Map 2
MARCH 23 1936 7 PM

of 1/2 unit and a 12-hourly movement of one unit. Considering the movement along the axis as a component of a rectangular resolution of the path of the center of the low we only need draw a perpendicular from the axis to the path of the pressure center. These distances along the direction arrow are marked by crosses for 6 and 12 hours. The actual low pressure center, from the data available, is marked for both the 6-hour and 12-hour movements by the letter "A" within a circle. It may be noticed that the actual velocity of the center is very slightly greater than the computed velocity, and slightly to the westward of the assumed path, that of the warm sector isobars. This differenc in the direction of the path should have been expected in view of the falling tendencies west of north of the center of the low.

Turning attention to the eastern cold front, difficulty may be expected in the use of the trough formula over Kansas, because of the nearness of the western front with the resulting crowding of high values of pressure tendencies immediately west of the front, and over southeastern Oklahoma because of lack of data. This makes the use of the trough formula doubtful. It is even more difficult in this, and most other cases, to determine the tendency values immediately behind the front for use in the front formula. There is another method in using the front formula that has not as yet been sufficientl tried. This is in the examination of the hourly reports along the airways for determining the pressure tendencies from the changes in pressure with the passage of a front. This method is used in computing the velocity of this front at three points. For the tendency values the hourly reports were examined at Lebo, Kansas, Tulsa, Oklahoma and Dallas, Texas. Of course, values for tendencies thus obtained will be from hourly differences in sea level pressures. Ordinarily, such a practice will not result in serious error. In all cases the pressure difference during the hour succeeding the passage was used as the pressure tendency to the rear of the front and the pressure difference during the hour preceeding the passage of the front as the tendency ahead of the front. During certain parts of the day this practice might reasonably be expected to result in error, due to diurnal changes in pressure.

At Lebo the pressure during the hour following the passage of the front seemed not entirely reliable; therefore, the accumulated rise for two hours after the passage of the front was used as the pressure tendency to the rear of the front +.12 inch. In this case as well as in the other two the pressure ahead of the front was stationary, in other words the tendency ahead of the front was zero. For the northern point, that is near Lebo, the pressure ascendant ahead of the front was computed as a mean of the distance from the front to the second isobar ahead of the front, and the pressure ascendant back of the front a mean of the distance from the front to the second isobar back of the front. This gave a pressure difference of .14 of an inch in both cases. Substituting these values in the front formula the distances computed for 6 and 12-hours are marked on the axis by a short normal line across the axis.

For the middle point the hourly observations at Tulsa, Oklahoma were used. During the hour following passage of the front there was a pressure ris of .03 of an inch. With zero as the tendency ahead of the front and the pressure ascendant taken as a mean from the front to the first isobar ahead of the front, and using the same distance back of the front solutions were again

obtained for 6 and 12-hours as indicated by the short line normal across the axis.

The same procedure was followed for the southern point using the Dallas hourly observations where pressure rise during the hour following the passage of the front was .02 of an inch. In this case the pressure ascendant was computed as the average as that occurring between the front and the second isobar ahead of the front, and using the same distance back of the front.

The two smoothed dashed lines ahead of the cold front indicate the computed positions of the front at the succeeding 6-hour and 12-hour intervals. The solid line indicates the actual position. It will be noticed that the 12-hour agreement is better than the 6-hour. At the southern point the actual velocity during the first 6 hours is small and increases considerably during the night to a point where the front at the end of 12 hours is ahead of the computed position, while at 6 hours it is considerably back of the computed position.

It is necessary that the tendency interval be considered in advancing the front. Pressure tendencies appearing on a map are for 3 hour intervals. In the method used for this front in two cases tendencies were for a 1 hour interval, and in the other for a 2 hour interval. The writer does not recommend this procedure at present because of insufficient trial, but does recommend such application for experimental purposes. In this case especially for a 12 hour interval the computations worked out very well.

For the western front, an axis at the southern part where there is maximum of curvature, the trough formula was applied even though it was necessary to use a small unit and the pressure gradient weak. In this case no half unit was used; the numerator in the trough formula then is one-half of the difference in pressure tendencies at the full unit. The quantities substituted in the trough formula are indicated along the axis. The arrow head indicates the computed movement, and the "A" within a circle indicates the actual movement. The agreement between the computed and actual movement for this type of front is surprising. In many of the cases in which this type of front moves southward accelerations are such as to make large differences between computed and actual movements.

Because of the isallobaric discontinuity at the front the trough formula could not be used for the eastward advance of this front, and in the absence of pressure readings at Waynoka, Oklahoma, and Anthony, Kansas it is not possible to determine accurate tendencies immediately ahead of or in back of this front. The application of the front formula then is impossible. In view of the slightly larger 3 hour tendencies back of this western cold front with respect to those ahead of the front over those in back of the eastern front with respect to those ahead of the front it might be safely assumed that its velocity will be slightly greater than the eastern front. This is what actually occurred.

ACCELERATION

The numerical solution of the acceleration equations has not been practical as yet because of the term Δ T which from a weather map is the 3-hourly change in the pressure tendency. Only recently the 3-hourly pressure change

was inaugurated on airway sequence reports. Often, even with 3-hourly pressure changes, the numerical solution of the acceleration equation will not be dependable because the magnitude of $(p_{102}\,(\Delta T^{\frac{1}{2},0} - \Delta T^{-\frac{1}{2},0})$ usually is so small as to be within the limits of error of the tendency values. However, with fronts moving fairly rapidly and a large isallobaric ascendant behind it, it may not be at all impossible to apply the equation of acceleration for troughs. Further experience is needed since the adoption of 3-hourly pressure tendencies along the airways before anything definite can be decided as to the practical value of the numerical solution of this equation. It is, however, of considerable importance in the south.